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# ***Note on the Development of an Algebraic Fraction.***

BY CAPT. P. A. MACMAHON, R. A.

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In the *American Journal of Mathematics*, Vol. V, No. 3, M. Faà de Bruno has considered the development, in ascending powers of  $x$ , of the algebraic fraction

$$\phi(x) = \frac{1}{1 + a_1x + a_2x^2 + \dots + a_nx^n},$$

and has obtained the coefficient of  $x^p$  in the form of a determinant.

His result may be simply obtained as follows.

For convenience I take the fraction to be

$$f(x) = \frac{1}{1 - a_1x + a_2x^2 - \dots + (-)^n a_nx^n}.$$

Let

$$\begin{aligned} F(y) &= y^n - a_1y^{n-1} + a_2y^{n-2} - \dots + (-)^n a_n \\ &= (y - \alpha)(y - \beta)(y - \gamma) \dots \end{aligned}$$

so that  $\alpha, \beta, \gamma \dots$  are the roots of the equation  $F(y) = 0$ ;

then 
$$\frac{1}{y - \alpha} = \frac{1}{y} + \frac{\alpha}{y^2} + \frac{\alpha^2}{y^3} + \dots$$

and 
$$\frac{1}{F(y)} = \frac{1}{y^n} + \frac{H_1}{y^{n+1}} + \frac{H_2}{y^{n+2}} + \dots + \frac{H_p}{y^{n+p}} + \dots$$

wherein  $H_p$  represents the sum of the homogeneous symmetric functions, of weight  $p$ , of the roots of the equation

$$F(y) = 0.$$

Write now  $y = \frac{1}{x}$  and divide both sides of the resulting equation by  $x^n$ , thus obtaining

$$\frac{1}{1 - a_1x + a_2x^2 - \dots + (-)^n a_nx^n} = 1 + H_1x + H_2x^2 + \dots + H_px^p + \dots$$

It is well known that

$$H_p = \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_p \\ 1 & a_1 & a_2 & \dots & a_{p-1} \\ 0 & 1 & a_1 & \dots & a_{p-2} \\ 0 & 0 & 1 & \dots & a_{p-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & a_1 \end{vmatrix}$$

which is equivalent to M. Faà de Bruno's result.

Now 
$$H_p = \Sigma (-)^{p+a_1+a_2+a_3+\dots} \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \dots)!}{\alpha_1! \alpha_2! \alpha_3! \dots} \alpha_1^{a_1} \alpha_2^{a_2} \alpha_3^{a_3} \dots$$

the summation extending to all integer, including zero, solutions of the equation

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = p;$$

consequently we have the result

$$= \sum_{p=0}^{\infty} \sum_{k=0}^p (-)^{p+k} \frac{k!}{\alpha_1! \alpha_2! \alpha_3! \dots} \alpha_1^{a_1} \alpha_2^{a_2} \alpha_3^{a_3} \dots x^p,$$

where

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \dots &= k, \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots &= p. \end{aligned}$$

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#### NOTE BY DR. FRANKLIN.

The general coefficient in the expansion of

$$\frac{1}{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}$$

is obviously given immediately in the form of a determinant by comparison of coefficients. If the required series is

$$b_0 + b_1x + b_2x^2 + \dots + b_nx^n,$$

we have

$$(a_0 + a_1x + a_2x^2 + \dots + a_nx^n)(b_0 + b_1x + b_2x^2 + \dots + b_nx^n) = 1,$$

whence

$$\begin{aligned} a_0 b_0 &= 1 \\ a_1 b_0 + a_0 b_1 &= 0 \\ a_2 b_0 + a_1 b_1 + a_0 b_2 &= 0, \text{ etc.}, \end{aligned}$$

whence, solving for the  $b$ 's,

$$b_p = (-)^p \left( \frac{1}{a_0} \right)^{p+1} \begin{vmatrix} a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \dots & \dots & \dots & \dots & \dots \\ a_p & a_{p-1} & \dots & \dots & a_0 \end{vmatrix}.$$

The above is so obvious that I have been in the habit of regarding it as the natural method of obtaining the value of  $H_p$ , whereas, in the preceding note, Captain MacMahon has reversed the process.